Lecture 17. Coordinate systems

Thm Given a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ of a vector space V, every vector $\vec{v}$ in V can be uniquely written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \text{ with } c_1, c_2, \cdots, c_n \in \mathbb{R}.$$

Note We can thus put each vector $\vec{v}$ into a matrix form

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

called the $\mathcal{B}$-coordinate vector of $\vec{v}$.

$*$ We can study vector spaces using matrices

e.g. $\mathbb{P}_n$: the space of all polynomials of degree at most n with the standard basis given by $1, t, \cdots, t^n$.

$\Longrightarrow$ Every polynomial $p(t)$ in $\mathbb{P}_n$ can be uniquely written as

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n \text{ with } a_0, a_1, \cdots, a_n \in \mathbb{R}.$$

and thus may be put into the standard coordinate vector

$$[p(t)] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$*$ Standard coordinate vectors for $\mathbb{P}_n$ have $n+1$ entries.

Ex Let $\mathcal{B}$ be a basis of $\mathbb{R}^2$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(1) Find the vector $\vec{u} \in \mathbb{R}^2$ with $\mathcal{B}$-coordinate vector

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Sol The given $\mathcal{B}$-coordinate vector yields

$$\vec{u} = 1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

(2) Find the $\mathcal{B}$-coordinate vector of

$$\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

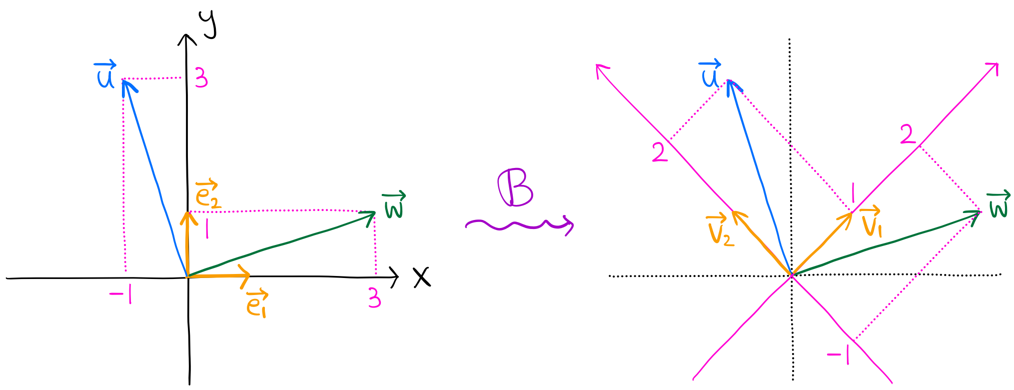
Sol Take A to be the matrix with columns $\vec{v}_1, \vec{v}_2$.

We want $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 \Longrightarrow \vec{w} = A\vec{x}$

$$\begin{array}{c} \left[\begin{array}{cc:c} 1 & -1 & 3 \\ 1 & 1 & 1 \end{array}\right] \\ \begin{array}{cc} A & \vec{w} \end{array} \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$\Longrightarrow x_1 = 2,\ x_2 = -1 \Longrightarrow \vec{w} = 2\vec{v}_1 - \vec{v}_2 \Longrightarrow [\vec{w}]_{\mathcal{B}} = \boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

Note Geometrically, $\mathcal{B}$-coordinates come from the axes along $\vec{v}_1, \vec{v}_2$.



Ex Determine whether each set of polynomials is a basis of $\mathbb{P}_2$.

(1) $p_1(t) = 2-t$, $p_2(t) = 1+t^2$, $p_3(t) = 3t-2t^2$

Sol Take A to be the matrix with columns $[p_1(t)], [p_2(t)], [p_3(t)]$.

The standard basis of $\mathbb{P}_2$ is given by $1, t, t^2$.

$$\Rightarrow \begin{cases} p_1(t) = 2\cdot 1 + (-1)\cdot t + 0\cdot t^2 \\ p_2(t) = 1\cdot 1 + 0\cdot t + 1\cdot t^2 \\ p_3(t) = 0\cdot 1 + 3\cdot t + (-2)\cdot t^2 \end{cases}$$

$$\Rightarrow [p_1(t)] = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad [p_2(t)] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_3(t)] = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \det(A) = -8 \neq 0$$

$\Rightarrow$ The set is $\boxed{\text{a basis of } \mathbb{P}_2}$

(2) $p_1(t) = 3+2t^2$, $p_2(t) = 2-t+t^2$, $p_3(t) = 1+t+t^2$

Sol Take A to be the matrix with columns $[p_1(t)], [p_2(t)], [p_3(t)]$.

The standard basis of $\mathbb{P}_2$ is given by $1, t, t^2$.

$$\Rightarrow \begin{cases} p_1(t) = 3\cdot 1 + 0\cdot t + 2\cdot t^2 \\ p_2(t) = 2\cdot 1 + (-1)\cdot t + 3\cdot t^2 \\ p_3(t) = 1\cdot 1 + 1\cdot t + 1\cdot t^2 \end{cases}$$

$$\Rightarrow [p_1(t)] = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad [p_2(t)] = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad [p_3(t)] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow \det(A) = 0$$

$\Rightarrow$ The set is $\boxed{\text{not a basis of } \mathbb{P}_2}$

Ex Let V be the subspace of $\mathbb{P}_3$ spanned by

$$p_1(t) = 2t+3t^2-t^3, \quad p_2(t) = 1-2t^2, \quad p_3(t) = 3+4t-2t^3$$

(1) Find a basis $\mathcal{B}$ of V.

Sol We find the coordinate vectors $[p_1(t)], [p_2(t)], [p_3(t)]$.

The standard basis of $\mathbb{P}_2$ is given by $1, t, t^2, t^3$.

$$\Rightarrow \begin{cases} p_1(t) = 0\cdot 1 + 2\cdot t + 3\cdot t^2 + (-1)\cdot t^3 \\ p_2(t) = 1\cdot 1 + 0\cdot t + (-2)\cdot t^2 + 0\cdot t^3 \\ p_3(t) = 3\cdot 1 + 4\cdot t + 0\cdot t^2 + (-2)\cdot t^3 \end{cases}$$

$$\Rightarrow [p_1(t)] = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad [p_2(t)] = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad [p_3(t)] = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -2 \end{bmatrix}$$

V is spanned by $p_1(t), p_2(t), p_3(t)$.

$\Rightarrow$ V may be regarded as $\mathrm{Col}(A)$ where A is the matrix with columns $[p_1(t)], [p_2(t)], [p_3(t)]$.

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 4 \\ 3 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \Rightarrow \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF(A) has leading 1's in column 1 and column 2.

$\Rightarrow$ A basis of $\mathrm{Col}(A)$ is given by $\underbrace{[p_1(t)], [p_2(t)]}_{\text{col 1, col 2 in } A}$

$\Rightarrow$ A basis of V is given by $\boxed{p_1(t), p_2(t)}$

(2) If possible, find the $\mathcal{B}$-coordinate vector of

$$p(t) = 2 + 6t + 5t^2 - 3t^3.$$

Sol We find the coordinate vector $[p(t)]$.

$$p(t) = 2\cdot 1 + 6\cdot t + 5\cdot t^2 + (-3)\cdot t^3 \Rightarrow [p(t)] = \begin{bmatrix} 2 \\ 6 \\ 5 \\ -3 \end{bmatrix}$$

Take B to be the matrix with columns $[p_1(t)], [p_2(t)]$.

We want $p(t) = x_1 p_1(t) + x_2 p_2(t) \Rightarrow [p(t)] = B\vec{x}$

$$\begin{array}{c} \left[\begin{array}{cc:c} 0 & 1 & 2 \\ 2 & 0 & 6 \\ 3 & -2 & 5 \\ -1 & 0 & -3 \end{array}\right] \\ \quad B \quad\ [p(t)] \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_1 = 3,\ x_2 = 2 \Rightarrow p(t) = 3p_1(t) + 2p_2(t) \Rightarrow [p(t)]_{\mathcal{B}} = \boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

Note If a polynomial does not lie in V, its $\mathcal{B}$-coordinate vector does not exist.

e.g. $q(t) = 1 - 3t + 2t^2 - t^3$

$\Rightarrow$ The equation $B\vec{x} = [q(t)]$ has no solutions.

$$\begin{array}{c} \left[\begin{array}{cc:c} 0 & 1 & 1 \\ 2 & 0 & -3 \\ 3 & -2 & 2 \\ -1 & 0 & -1 \end{array}\right] \\ \quad B \quad\ [q(t)] \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{a leading 1 in the last column}$$